Construction of particular solutions to nonlinear equations of Mathematical Physics by using matrix algebraic equation

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Abstract

The paper develops the method for construction of the families of particular solutions to the nonlinear Partial Differential Equations (PDE) without relation to the complete integrability. Method is based on the specific link between algebraic matrix equations and PDE. Example of (2+2)-dimensional generalization of Burgers equation is given.

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1 Introduction

Analysis of nonlinear Partial Differential Equations (PDE) is severe problem in mathematical physics. Many different methods have been developed for analytical investigation of nonlinear PDE during last decades: Inverse Scattering Problem [1, 2, 3, 4, 5, 6, 7], Sato theory [8, 9, 10, 11], Hirota bilinear method [12, 13, 14, 15], Penlevé method [16, 17, 18], $\bar{\partial}$-problem [19, 20, 21], with some generalizations [22, 23, 24, 25, 26], nonlinear $\bar{\partial}$-problem [27, 28, 29], recent modification of the dressing method based on the algebraic matrix equation [30, 31, 32]. A wide class of PDE (so-called completely integrable systems) has been studied better then others. Nevertheless, there are many methods which work in nonintegrable case as well: [12, 13, 14, 15, 16, 17, 18, 30, 31, 32].

We represent the method for construction of families of particular solutions to multidimensional nonlinear PDE without relation to the complete integrability. This method is based on general properties of linear algebraic matrix equations. Essentially we develop some ideas represented in the ref.[8] and recently in the ref. [11].

General algorithm is discussed in the Sec. 2. Sec. 3 deals with example of (2+2)-dimensional system of PDE together with family of particular solutions. By using an appropriate reduction, this system leads to Burgers equation. This example is simple demonstration of the method. Conclusions are given in the Sec.4.

2 General results

The algorithm represented in this section is based on the fundamental properties of linear matrix algebraic equation, which is written in the following form:

$$\Psi U = \Phi, \tag{1}$$

where $\Psi = \{\psi_{ij}\}$ is $N \times N$ matrix $U = [u_1, \dots, u_N]^T$, $\Phi = [\phi_1, \dots, \phi_N]^T$. Let us recall these properties.

1. If Ψ is nondegenerate matrix, i.e.

$$\det \Psi \neq 0, \tag{2}$$

then equation (1) has unique solution which can be written in the following form:

$$U = \Psi^{-1}\Phi. \tag{3}$$

Only nondegenerate matrices Ψ will be considered hereafter.

2. (*consequence of the previous property*) If $\Phi = 0$ and condition (2) is held, then the equation (1) has only the trivial solution

$$U \equiv 0. \tag{4}$$

3. (*superposition principle*) Consider the set of K matrix equations with the same matrix Ψ:

$$\Psi U_i = \Phi_i, \quad i = 1, \dots, K. \tag{5}$$

Then for any set of scalars b_k $(k = 1, \dots, K)$, function $\tilde{U} = \sum_{k=1}^{K} b_k U_k$ is solution of the following matrix equation

$$\Psi \tilde{U} = \sum_{k=1}^{K} b_k \Phi_k. \tag{6}$$

4. (*consequence of properties 2 and 3*) If columns Φ_i are linearly dependent, i.e there are scalars a_k, $k = 1, \dots, K$, such that

$$\sum_{k=1}^{K} a_k \Phi_k = 0, \tag{7}$$

then

$$\sum_{i=1}^{K} a_i U_i = 0. \tag{8}$$

Note that analogous properties of linear integral equation have been used in the classical dressing method based on the $\bar{\partial}$-problem [19, 20].

To relate the matrix equation (1) with the system of nonlinear PDE, one needs to introduce the set of additional variables $\mathbf{x} = (x_1, \dots, x_M)$ in

matrices Ψ, Φ and U, which are independent variables of succeeding system of nonlinear PDE. M is dimension of $\mathbf{x}$-space. For this purpose we use the following system of linear differential equations

$$M_k\Phi + N_k\Psi = \Psi A_k, \quad k = 1, \ldots, P, \tag{9}$$

where M_k and N_k are linear differential operators with matrix coefficients; A_k are $N \times N$ matrices, which are constant for the sake of simplicity. The number of these equations, P, depends on situation. Equations (9) introduce variables $\mathbf{x}$ into the matrices Ψ and Φ. Due to the eq.(3), elements u_i of the column U are functions of variables $\mathbf{x}$ as well.

Operators M_k and N_k may not be arbitrary. They have to satisfy two conditions:

1. Overdetermined system (9) is compatible.

2. Operators M_k and N_k have to provide existence of differential operator $\mathcal{M}$ with *non-constant* matrix coefficients such that

$$\begin{aligned} \mathcal{M}(\Phi - \Psi W) &= \sum_{k=1}^{P} (M_k\Phi + N_k\Psi)\tilde{U}_k + \Psi\tilde{V} = \Psi B(U) \equiv 0, \\ B(U) &= \sum_{k=1}^{P} A_k\tilde{U}_k + \tilde{V}, \end{aligned} \tag{10}$$

   where $\tilde{U}_k$ and $\tilde{V}$ are some matrix functions of elements u_i and their derivatives (see eqs.(23-28) as an example).

Since $\det\Psi \neq 0$, the last equation means:

$$B(U) = 0, \tag{11}$$

which represents the system of nonlinear PDE on functions u_i. The structure of this system is defined by the equations (9).

One can see that the matrix equation (1) is generalization of the linear systems used in the refs.[8, 11]. In fact, let us introduce notations

$$\phi_i \equiv \phi^{(i)}, \quad \psi_{ij} \equiv \psi_j^{(i)}, \tag{12}$$

and rewrite the matrix equation (1) in the form of system of N equations:

$$\phi^{(i)} = \sum_{j=1}^{N} u_j \psi_j^{(i)}, \quad i = 1, \ldots, N. \tag{13}$$

Two evident reductions are possible (these reductions will not be used in the succeeding sections).

1. If

$$\psi_j^{(i)} = \partial_{x_1}^j \psi^{(i)}, \quad \phi^{(i)} = \partial_{x_1}^{(N+1)} \psi^{(i)}, \quad i, j = 1, \ldots, N, \tag{14}$$

then system (13) is system of N ordinary differential equations on the function $\psi^{(i)}$ which is basic for the algorithm represented in the ref.[8].

2. Introduce multi-index $\beta = (\beta_1, \ldots, \beta_M)$, $||\beta|| = \sum_{k=1}^{M} \beta_k$ instead of index j (M is dimension of $\mathbf{x}$-space), and use notations

$$\psi_\beta^{(i)} = \prod_{k=1}^{M} \partial_{x_k}^{\beta_k} \psi^{(i)} \equiv \partial^\beta \psi^{(i)}, \quad ||\beta|| = 1, \ldots, N, \tag{15}$$

$$\phi^{(i)} = \partial^\alpha \psi^{(i)}, \quad ||\alpha|| = N + 1. \tag{16}$$

Then the system (13) can be written in the form

$$\partial^\alpha \psi^{(i)} = \sum_{||\beta||=0}^{N} u_\beta \partial^\beta \psi^{(i)}, \quad i = 1, \ldots, Q, \tag{17}$$

where Q is the number of terms in the right hand side of the eq.(17), α takes any value such that $||\alpha|| = N + 1$. Eq.(17) is basic in the ref.[11].

Hereafter, we will use eq. (13) instead of matrix equation (1). Superscript i runs the values from 1 to N, unless otherwise specified.

3 Example: (2+2)-dimensional generalization of Burgers equation

In this section we consider (2+2)-dimensional equations with following notations for independent variables:

$$x \equiv x_1, \quad y \equiv x_2, \quad t \equiv x_3, \quad \tau \equiv x_4. \tag{18}$$

For convenience of construction, we separate the first term of the sum in the eq. (13):

$$\phi^{(i)} = u_1\psi_1^{(i)} + \sum_{k=2}^{N} u_k\psi_k^{(i)}. \tag{19}$$

Introduce two differential operators L_1 and L_2:

$$\begin{aligned} L_1 &= l_1 + \delta_{11}u_{1x} + \delta_{12}u_{1y}, \quad l_1 = \partial_t + \partial_x^2 + \gamma_1\partial_x\partial_y, \\ L_2 &= l_2 + \delta_{21}u_{1x} + \delta_{22}u_{1y}, \quad l_2 = \partial_\tau + \partial_y^2 + \gamma_2\partial_x\partial_y. \end{aligned} \tag{20}$$

Let us apply operator L_1 to the eq.(19) and investigate the result

$$\begin{aligned} & -l_1\phi^{(i)} - \delta_{11}u_{1x}\phi^{(i)} - \delta_{12}u_{1y}\phi^{(i)} + (L_1u_1)\psi_1^{(i)} + u_1(l_1\psi_1^{(i)}) + \\ & \qquad u_{1x}(2\psi_1^{(i)}{}_x + \gamma_1\psi_1^{(i)}{}_y) + \gamma_1u_{1y}\psi_1^{(i)}{}_x + \\ & \sum_{k=1}^{N}\Big((L_1u_k)\psi_k^{(i)} + u_k(l_1\psi_k^{(i)}) + u_{kx}(2\psi_k^{(i)}{}_x + \gamma_1\psi_k^{(i)}{}_y) + \gamma_1u_{ky}\psi_k^{(i)}{}_x\Big) = 0 \end{aligned} \tag{21}$$

Our purpose is to represent the equation (21) in the form of linear homogeneous equation

$$\sum_{j=1}^{N} F_{1k}\psi_k^{(i)} = 0, \tag{22}$$

where F_{1k} are some expressions of u_k and their derivatives. Owing to the property 4 (eqs.(7) and (8)), this would mean that $F_{1k} \equiv 0$, $k = 1, \ldots, N$. The simplest way to result in the eq. (22) is introduction of the system of differential equations (9) on the functions $\psi_j^{(i)}$ and $\phi^{(i)}$, which looks as follows:

$$\phi^{(i)} - \frac{1}{\delta_{11}}(2\psi_1^{(i)}{}_x + \gamma_1\psi_1^{(i)}{}_y) = \sum_{k=2}^{N}\tilde{\alpha}_{1k}\psi_k^{(i)}, \tag{23}$$

$$\phi^{(i)} - \frac{\gamma_1}{\delta_{12}}\psi_1^{(i)}{}_x = \sum_{k=2}^{N}\alpha_{1k}\psi_k^{(i)}, \tag{24}$$

$$\psi^{(i)}_{2\ x} = \beta_1\psi^{(i)}_1 + \sum_{k=2}^{N}\alpha_{3k}\psi^{(i)}_k, \ \ \psi^{(i)}_{2\ y} = \beta_2\psi^{(i)}_1 + \sum_{k=2}^{N}\alpha_{4k}\psi^{(i)}_k, \tag{25}$$

$$\psi^{(i)}_{j\ x} = \sum_{k=2}^{N}\alpha_{(2j-1)k}\psi^{(i)}_k, \ \ \psi^{(i)}_{j\ y} = \sum_{k=2}^{N}\alpha_{(2j)k}\psi^{(i)}_k, \ \ j > 2, \tag{26}$$

$$l_1\psi^{(i)}_j = \sum_{k=2}^{N}\beta_{1jk}\psi^{(i)}_k, \ \ l_2\psi^{(i)}_j = \sum_{k=2}^{N}\beta_{2jk}\psi^{(i)}_k, \ \ j \geq 1, \tag{27}$$

$$l_1\phi^{(i)} = \sum_{k=2}^{N}\beta_{10k}\psi^{(i)}_k, \ \ l_2\phi^{(i)} = \sum_{k=2}^{N}\beta_{20k}\psi^{(i)}_k. \tag{28}$$

Here parameters $\tilde{\alpha}_{ij}, \alpha_{ij}, \beta_i, \beta_{ijk}$ have to provide compatibility of the overdetermined system (23)-(28). Equation (23) can be simplified if one eliminates $\phi^{(i)}$ by using the eq.(24):

$$\psi^{(i)}_{1\ y} - G\psi^{(i)}_{1\ x} = \sum_{k=2}^{N}\alpha_{2k}\psi^{(i)}_k, \ \ G = \frac{\delta_{11}}{\delta_{12}} - \frac{2}{\gamma_1}, \ \ \alpha_{2k} = \frac{\delta_{11}}{\gamma_1}(\alpha_{1k} - \tilde{\alpha}_{1k}). \tag{29}$$

The system (24)-(29) will be used below.

Now the equation (21) takes the form (22) with

$$\begin{aligned} F_{11} = u_{1t} + u_{1xx} + \gamma_1 u_{1xy} + \delta_{11}u_1u_{1x} + \delta_{12}u_1u_{1y} + \\ (2\beta_1 + \beta_2\gamma_1)u_{2x} + \beta_1\gamma_1 u_{2y} = 0. \end{aligned} \tag{30}$$

Expressions for F_{1k} with $k > 1$ will not be used.

Analogously, apply operator L_2 to the eq.(19):

$$\begin{aligned} -l_2\phi^{(i)} - \delta_{21}u_{1x}\phi^{(i)} - \delta_{22}u_{1y}\phi^{(i)} + (L_2u_1)\psi^{(i)}_1 + u_1(l_2\psi^{(i)}_1) + \\ u_{1y}(2\psi^{(i)}_{1\ y} + \gamma_2\psi^{(i)}_{1\ x}) + \gamma_2 u_{1x}\psi^{(i)}_{1\ y} + \\ \sum_{k=1}^{N}\Big((L_2u_k)\psi^{(i)}_k + u_k(l_2\psi^{(i)}_k) + u_{ky}(2\psi^{(i)}_{k\ y} + \gamma_2\psi^{(i)}_{k\ x}) + \gamma_2 u_{kx}\psi^{(i)}_{k\ y}\Big) = 0 \end{aligned} \tag{31}$$

Due to the relations (24)-(29) one gets analogous expression

$$\sum_{j=1}^{N} F_{2k}\psi^{(i)}_k = 0, \tag{32}$$

if only

$$\delta_{21} = \frac{\gamma_2}{\gamma_1^2}(\delta_{11}\gamma_1 - 2\delta_{12}), \quad \delta_{22} = \frac{1}{\gamma_1^2}(\delta_{12}\gamma_1\gamma_2 + 2\delta_{11}\gamma_1 - 4\delta_{12}). \tag{33}$$

Coefficients F_{2k} are expressed in terms of the functions u_k and their derivatives with

$$\begin{aligned} F_{21} = u_{1\tau} + u_{1yy} + \gamma_2 u_{1xy} + \frac{\gamma_2}{\gamma_1^2}(\delta_{11}\gamma_1 - 2\delta_{12})u_1 u_{1x} + & \quad (34) \\ \frac{1}{\gamma_1^2}(\delta_{12}\gamma_1\gamma_2 + 2\delta_{11}\gamma_1 - 4\delta_{12})u_1 u_{1y} + \beta_2\gamma_2 u_{2x} + (2\beta_2 + \beta_1\gamma_2)u_{2y} &= 0, \end{aligned}$$

Eqs.(30) and (34) form the system of two nonlinear PDE on the functions u_1 and u_2.

One shell discuss the structure of the system (24-29). Operators L_1 and L_2, given by the formulas (20), determine the left hand sides of the equations in this system. The structure of the system of nonlinear PDE is defined by the operators L_1 and L_2 and by the right hand sides of the equations in the system (24-29). Only terms proportional to $\psi_1^{(i)}$ effect on the structure of the nonlinear system of PDE. At least one of the equations (24-29) must have this term (see eqs.(25)). Otherwise, only simplest solutions to the nonlinear system of PDE are available. Other terms enrich the family of available particular solutions.

Function $\phi^{(i)}$ may not be linear combination of functions $\psi_j^{(i)}$ (see eqs. (23,24)). Otherwise all coefficients u_j in eq.(19) are zero. Equations (27) and (28) introduce dependence on variables t and τ. Right hand sides of these equations are zero in the examples below for the sake of simplicity.

3.1 Construction of particular solutions

This section is devoted to construction of some families of particular solutions to the system (30,34). We will use simplified form of the system (24-29):

$$\phi^{(i)} = \frac{\gamma_1}{\delta_{12}}\psi^{(i)}_{1\ x} + \alpha_{12}\psi_2^{(i)}, \tag{35}$$

$$\psi^{(i)}_{1\ y} = G\psi^{(i)}_{1\ x}, \quad G = \frac{\delta_{11}}{\delta_{12}} - \frac{2}{\gamma_1}, \tag{36}$$

$$\psi^{(i)}_{2\ x} = \beta_1\psi_1^{(i)} + \alpha_{32}\psi_2^{(i)}, \quad \psi^{(i)}_{2\ y} = \beta_2\psi_1^{(i)} + \alpha_{42}\psi_2^{(i)}, \tag{37}$$

$$
\begin{aligned}
{\psi_j^{(i)}}_x &= \sum_{k=2}^N \alpha_{(2j-1)k}\psi_k^{(i)}, \;\; {\psi_j^{(i)}}_y = \sum_{k=2}^N \alpha_{(2j)k}\psi_k^{(i)}, \;\; j>2, & (38)\\
l_k\psi_j^{(i)} &= 0, \;\; l_k\phi^{(i)} = 0, \;\; k=1,2, \;\; j=1,\dots,N & (39)
\end{aligned}
$$

Below are two examples with $N = 2$ and $N = 4$. Example with $N = 3$ does not have a principal difference with the case $N = 2$. Solutions from both families considered below are parameterized by the arbitrary functions of single variable.

3.1.1 One-dimensional kink

Let $N = 2$, superscript i takes the values 1 and 2. The system (19) has the following form:

$$
\phi^{(i)} = u_1\psi_1^{(i)} + u_2\psi_2^{(i)}, \tag{40}
$$

with

$$
u_1 = \frac{\triangle_1}{\triangle}, \;\; u_2 = \frac{\triangle_2}{\triangle}, \tag{41}
$$

$$
\triangle = \begin{vmatrix} \psi_1^{(1)} & \psi_2^{(1)} \\ \psi_1^{(2)} & \psi_2^{(2)} \end{vmatrix}, \;\; \triangle_1 = \begin{vmatrix} \phi^{(1)} & \psi_2^{(1)} \\ \phi^{(2)} & \psi_2^{(2)} \end{vmatrix}, \;\; \triangle_2 = \begin{vmatrix} \psi_1^{(1)} & \phi^{(1)} \\ \psi_1^{(2)} & \phi^{(2)} \end{vmatrix}, \tag{42}
$$

We need only equations (35)-(37) and (39) (with $j = 1, 2$) to find functions $\psi_j^{(i)}$ $(j = 1, 2)$ and $\phi^{(i)}$. They admit the following solutions

$$
\begin{aligned}
\psi_1^{(i)} &= \int c_1^{(i)}(k)e^{kx+qy+\omega t+\nu\tau}dk, & (43)\\
\psi_2^{(i)} &= \int c_2^{(i)}(k)e^{kx+qy+\omega t+\nu\tau}dk, & (44)\\
\phi^{(i)} &= \int c^{(i)}(k)e^{kx+qy+\omega t+\nu\tau}dk, & (45)\\
c^{(i)}(k) &= \frac{c_2^{(i)}(k)}{\beta_1\delta_{12}}(\gamma_1k^2 - \alpha_{32}\gamma_1k + \alpha_{12}\beta_1\delta_{12}), & (46)\\
c_1^{(i)}(k) &= \frac{c_2^{(i)}(k)}{\beta_1}(k-\alpha_{32}), & (47)
\end{aligned}
$$

$$q = \frac{\beta_2}{\beta_1} k, \tag{48}$$

$$\omega = -k^2 - kq\gamma_1 = -\frac{k^2}{\beta_1}(\beta_1 + \beta_2\gamma_1), \tag{49}$$

$$\nu = -q^2 - kq\gamma_2 = -\frac{k^2\beta_2}{\beta_1^2}(\beta_2 + \beta_1\gamma_2), \tag{50}$$

$$\delta_{11} = \frac{\delta_{12}}{\beta_1\gamma_1}(\beta_2\gamma_1 + 2\beta_1), \quad \alpha_{42} = \frac{\beta_2}{\beta_1}\alpha_{32} \tag{51}$$

One can see from the eqs.(43-47) that functions $\psi_2^{(i)}$ are arbitrary functions of single variable (say, variable x). So that appropriate solutions (41) depend on two arbitrary functions of single variable.

As a simple example, let us take the following expressions for the functions $\psi_j^{(i)}$ and $\phi^{(i)}$:

$$\psi_1^{(1)} = \frac{k_1 - \alpha_{32}}{\beta_1} s_1 E_1, \tag{52}$$

$$\psi_1^{(2)} = \frac{k_2 - \alpha_{32}}{\beta_1} s_2 E_2 + \frac{k_3 - \alpha_{32}}{\beta_1} s_3 E_3, \tag{53}$$

$$\psi_2^{(1)} = s_1 E_1, \tag{54}$$

$$\psi_2^{(2)} = s_2 E_2 + s_3 E_3, \tag{55}$$

$$\phi^{(1)} = \frac{s_1 E_1}{\beta_1 \delta_{12}}(\alpha_{12}\beta_1\delta_{12} - \alpha_{32}\gamma_1 k_1 + \gamma_1 k_1^2), \tag{56}$$

$$\begin{aligned} \phi^{(2)} = {} & \frac{s_2 E_2}{\beta_1 \delta_{12}}(\alpha_{12}\beta_1\delta_{12} - \alpha_{32}\gamma_1 k_2 + \gamma_1 k_2^2) + \\ & \frac{s_3 E_3}{\beta_1 \delta_{12}}(\alpha_{12}\beta_1\delta_{12} - \alpha_{32}\gamma_1 k_3 + \gamma_1 k_3^2), \\ & E_n = e^{k_n x + q_n y + \omega_n t + \nu_n \tau}, \quad n = 1, 2, 3, \end{aligned} \tag{57}$$

where

$$q_n = \frac{\beta_2}{\beta_1} k_n, \quad \omega_n = -\frac{k_n^2}{\beta_1}(\beta_1 + \beta_2\gamma_1), \quad \nu_n = -\frac{k_n^2\beta_2}{\beta_1^2}(\beta_2 + \beta_1\gamma_2), \quad n = 1, 2, 3, \tag{58}$$

which are in agreement with eqs.(43)-(51). Determinants (42) are expressed

as follows:

$$\begin{aligned}
\triangle &= \frac{s_1E_1}{\beta_1}((k_1-k_2)s_2E_2+(k_1-k_3)s_3E_3), && (59)\\
\triangle_1 &= -\frac{\gamma_1 s_1E_1}{\beta_1\delta_{12}}((k_1-k_2)(\alpha_{32}-k_1-k_2)s_2E_2+ && (60)\\
&\quad (k_1-k_3)(\alpha_{32}-k_1-k_3)s_3E_3),\\
\triangle_2 &= \frac{s_1E_1}{\beta_1^2\delta_{12}}\times && (61)\\
&\quad ((k_1-k_2)(\alpha_{12}\beta_1\delta_{12}-\alpha_{32}^2\gamma_1+\alpha_{32}\gamma_1(k_1+k_2)-\gamma_1k_1k_2)s_2E_2+\\
&\quad (k_1-k_3)(\alpha_{12}\beta_1\delta_{12}-\alpha_{32}^2\gamma_1+\alpha_{32}\gamma_1(k_1+k_3)-\gamma_1k_1k_3)s_3E_3).
\end{aligned}$$

Functions u_1 and u_2 have no singularities provided that $\triangle \neq 0$ for all values of parameters x, y, t, τ, which happens if only

$$\mathrm{sign}((k_1-k_2)s_2) = \mathrm{sign}((k_1-k_3)s_3). \qquad (62)$$

Taking into account relations (51), system (30,34) gets the following form:

$$\begin{aligned}
u_{1t}+u_{1xx}+\gamma_1u_{1xy}+\frac{\delta_{12}}{\beta_1\gamma_1}(2\beta_1+\beta_2\gamma_1)u_1u_{1x}+ & && (63)\\
\delta_{12}u_1u_{1y}+(2\beta_1+\beta_2\gamma_1)u_{2x}+\beta_1\gamma_1u_{2y} &= 0,\\
u_{1\tau}+u_{1yy}+\gamma_2u_{1xy}+\frac{\gamma_2\delta_{12}\beta_2}{\gamma_1\beta_1}u_1u_{1x}+\frac{\delta_{12}}{\gamma_1\beta_1}(2\beta_2+\beta_1\gamma_2)u_1u_{1y}+ & && (64)\\
\beta_2\gamma_2u_{2x}+(2\beta_2+\beta_1\gamma_2)u_{2y} &= 0,
\end{aligned}$$

Note that owing to the relations (48-50) functions u_1 and u_2 depend on two variables rather then four:

$$X = x+\frac{\beta_2}{\beta_1}y,\quad T = \frac{\beta_1+\beta_2\gamma_1}{\beta_1}t+\frac{\beta_2(\beta_2+\beta_1\gamma_2)}{\beta_1^2}\tau \qquad (65)$$

Because of this fact, functions u_1 and u_2, given by the formulas (41,59-61), represent single kink, which is essentially one-dimensional. Moreover, in virtue of eqs.(65), the above system (63,64) can be written in the form of differentiated Burghers equation on the function u_1 with independent variables X and T.

3.1.2 Multidimensional kinks

In this section $N = 4$, superscript i takes values from 1 to 4, unless otherwise specified. The system (19) has the form

$$\phi^{(i)} = u_1\psi_1^{(i)} + u_2\psi_2^{(i)} + u_3\psi_3^{(i)} + u_4\psi_4^{(i)}. \tag{66}$$

Formulas (41) for the functions u_1 and u_2 are held with determinants in the following form:

$$\triangle = \begin{vmatrix} \psi_1^{(1)} & \psi_2^{(1)} & \psi_3^{(1)} & \psi_4^{(1)} \\ \psi_1^{(2)} & \psi_2^{(2)} & \psi_3^{(2)} & \psi_4^{(2)} \\ \psi_1^{(3)} & \psi_2^{(3)} & \psi_3^{(3)} & \psi_4^{(3)} \\ \psi_1^{(4)} & \psi_2^{(4)} & \psi_3^{(4)} & \psi_4^{(4)} \end{vmatrix}, \quad \triangle_1 = \begin{vmatrix} \phi^{(1)} & \psi_2^{(1)} & \psi_3^{(1)} & \psi_4^{(1)} \\ \phi^{(2)} & \psi_2^{(2)} & \psi_3^{(2)} & \psi_4^{(2)} \\ \phi^{(3)} & \psi_2^{(3)} & \psi_3^{(3)} & \psi_4^{(3)} \\ \phi^{(4)} & \psi_2^{(4)} & \psi_3^{(4)} & \psi_4^{(4)} \end{vmatrix}, \tag{67}$$

$$\triangle_2 = \begin{vmatrix} \psi_1^{(1)} & \phi^{(1)} & \psi_3^{(1)} & \psi_4^{(1)} \\ \psi_1^{(2)} & \phi^{(2)} & \psi_3^{(2)} & \psi_4^{(2)} \\ \psi_1^{(3)} & \phi^{(3)} & \psi_3^{(3)} & \psi_4^{(3)} \\ \psi_1^{(4)} & \phi^{(4)} & \psi_3^{(4)} & \psi_4^{(4)} \end{vmatrix}$$

Functions $\psi_1^{(i)}$, $\psi_2^{(i)}$, $\phi^{(i)}$ are defined by the same formulas (35-37) and (39) ($j = 1, 2$) together with solutions (43)-(51). Because of this, solution (41) may depend on four arbitrary functions of single variable (compare with the paragraph below the eq.(51)).

Functions $\psi_3^{(i)}$ and $\psi_4^{(i)}$ are solutions of the equations (38,39) with $j = 3, 4$:

$$\psi_3^{(i)}{}_x = \alpha_{53}\psi_3^{(i)} + \alpha_{54}\psi_4^{(i)} + \alpha_{52}\psi_2^{(i)}, \quad \psi_3^{(i)}{}_y = \alpha_{63}\psi_3^{(i)} + \alpha_{64}\psi_4^{(i)} + \alpha_{62}\psi_2^{(i)}, \tag{68}$$

$$\psi_4^{(i)}{}_x = \alpha_{73}\psi_3^{(i)} + \alpha_{74}\psi_4^{(i)} + \alpha_{72}\psi_2^{(i)}, \quad \psi_4^{(i)}{}_y = \alpha_{83}\psi_3^{(i)} + \alpha_{84}\psi_4^{(i)} + \alpha_{82}\psi_2^{(i)}, \tag{69}$$

$$l_k\psi_j^{(i)} = 0, \quad j = 3, 4, \quad k = 1, 2. \tag{70}$$

This is a system of linear nonhomogeneous equations with solution in the following general form:

$$\psi_3^{(i)} = \psi_{30}^{(i)} + \psi_{3p}^{(i)}, \quad \psi_4^{(i)} = \psi_{40}^{(i)} + \psi_{4p}^{(i)}, \tag{71}$$

where $\psi_{j0}^{(i)}$ and $\psi_{jp}^{(i)}$ are general solution of homogeneous system, related with the system (68-70), and particular solution of nonhomogeneous system (68-70) respectively.

The general solution of homogeneous system reads:

$$
\begin{aligned}
\psi_{30}^{(i)} &= p_1^{(i)} r_3^{(1)} e^{\tilde{k}_1 x+\tilde{q}_1 y+\tilde{\omega}_1 t+\tilde{\nu}_1\tau} + p_2^{(i)} r_3^{(2)} e^{\tilde{k}_2 x+\tilde{q}_2 y+\tilde{\omega}_2 t+\tilde{\nu}_2\tau}, && (72)\\
\psi_{40}^{(i)} &= p_1^{(i)} r_4^{(1)} e^{\tilde{k}_1 x+\tilde{q}_1 y+\tilde{\omega}_1 t+\tilde{\nu}_1\tau} + p_2^{(i)} r_4^{(2)} e^{\tilde{k}_2 x+\tilde{q}_2 y+\tilde{\omega}_2 t+\tilde{\nu}_2\tau}, && (73)\\
r_4^{(i)} &= \frac{(\tilde{k}_i-\alpha_{53})r_3^{(i)}}{\alpha_{54}}, && (74)\\
\tilde{q}_j &= \frac{1}{\alpha_{54}}(\alpha_{74}\tilde{k}_j+\alpha_{54}\alpha_{73}-\alpha_{53}\alpha_{74}), \quad j=1,2, && (75)\\
\tilde{k}_{1,2} &= \frac{1}{2}\Big(\alpha_{53}+\alpha_{64}\pm && (76)\\
&\quad \frac{1}{\sqrt{\alpha_{74}}}\sqrt{\alpha_{53}^2\alpha_{74}-2\alpha_{53}\alpha_{64}\alpha_{74}+\alpha_{64}^2\alpha_{74}+4\alpha_{54}^2\alpha_{83}}\Big),
\end{aligned}
$$

where $\tilde{\omega}_j$ and $\tilde{\nu}_j$ are related with $\tilde{k}_j$ and $\tilde{q}_j$ by the equations

$$
\tilde{\omega}_j = -\tilde{k}_j^2 - \tilde{k}_j\tilde{q}_j\gamma_1, \quad \tilde{\nu}_j = -\tilde{q}_j^2 - \tilde{k}_j\tilde{q}_j\gamma_2, \tag{77}
$$

which follows from the eqs.(70).

The particular solution can be taken in the form (in virtue of the eq.(44))

$$
\begin{aligned}
\psi_{3p}^{(i)} &= \int \Gamma_1^{(i)}(k)c_2^{(i)}(k)\exp((x+\beta_2/\beta_1 y)k+\omega(k)t+\nu(k)\tau)dk, && (78)\\
\psi_{4p}^{(i)} &= \int \Gamma_2^{(i)}(k)c_2^{(i)}(k)\exp((x+\beta_2/\beta_1 y)k+\omega(k)t+\nu(k)\tau)dk,
\end{aligned}
$$

with $\omega(k)$ and $\nu(k)$ given by the eq.(49,50). Substitution of the eqs.(78) into the eqs.(68,69) results in the following expressions for Γ_1 and Γ_2 together with additional relations among parameters α_{ij}:

$$
\begin{aligned}
\Gamma_1(k) &= \frac{\alpha_{52}^2}{\alpha_{52}k-\alpha_{52}\alpha_{53}-\alpha_{54}\alpha_{62}}, \quad \Gamma_2(k) = \frac{\alpha_{62}}{\alpha_{52}}\Gamma_1(k), && (79)\\
\alpha_{52} &= \frac{\alpha_{54}\alpha_{62}(\alpha_{54}\beta_2-\alpha_{74}\beta_1)}{(\alpha_{53}\alpha_{74}\beta_1-\alpha_{64}\alpha_{74}\beta_1+\alpha_{54}\alpha_{84}\beta_1-\alpha_{53}\alpha_{54}\beta_2)},\\
\alpha_{72} &= \frac{\alpha_{52}\beta_2}{\beta_1}, \quad \alpha_{82} = \frac{\alpha_{62}\beta_2}{\beta_1}, && (80)\\
\alpha_{83} &= \frac{\alpha_{74}(\alpha_{84}\beta_1-\alpha_{74}\beta_2)}{\alpha_{54}(\alpha_{54}\beta_2-\alpha_{74}\beta_1)^2}\times && (81)\\
&\quad (\alpha_{53}\alpha_{74}\beta_1-\alpha_{64}\alpha_{74}\beta_1+\alpha_{54}\alpha_{84}\beta_1-\alpha_{53}\alpha_{54}\beta_2)
\end{aligned}
$$

As an example, let us choose the following expressions for $\psi_j^{(i)}$ and $\phi^{(i)}$:

$$\psi_1^{(i)} = s_i c_1^{(i)} e^{k_i x + q_i y + \omega_i t + \nu_i \tau}, \quad i = 1, 2, 3, \tag{82}$$
$$\psi_1^{(4)} = s_4 c_1^{(4)} e^{k_4 x + q_4 y + \omega_4 t + \nu_4 \tau} + s_5 c_1^{(5)} e^{k_5 x + q_5 y + \omega_5 t + \nu_5 \tau}, \tag{83}$$
$$\psi_2^{(i)} = s_i e^{k_i x + q_i y + \omega_i t + \nu_i \tau}, \quad i = 1, 2, 3, \tag{84}$$
$$\psi_2^{(4)} = s_4 e^{k_4 x + q_4 y + \omega_4 t + \nu_4 \tau} + s_5 e^{k_5 x + q_5 y + \omega_5 t + \nu_5 \tau}, \tag{85}$$
$$\phi^{(i)} = s_i c^{(i)} e^{k_i x + q_i y + \omega_i t + \nu_i \tau}, \quad i = 1, 2, 3, \tag{86}$$
$$\phi^{(4)} = s_4 c^{(4)} e^{k_4 x + q_4 y + \omega_4 t + \nu_4 \tau} + s_5 c^{(5)} e^{k_5 x + q_5 y + \omega_5 t + \nu_5 \tau}, \tag{87}$$
$$\psi_3^{(1)} = p_1 e^{\tilde{k}_1 x + \tilde{q}_1 y + \tilde{\omega}_1 t + \tilde{\nu}_1 \tau} + \Gamma_1(k_1) s_1 e^{k_1 x + q_1 y + \omega_1 t + \nu_1 \tau}, \tag{88}$$
$$\psi_3^{(2)} = p_2 e^{\tilde{k}_2 x + \tilde{q}_2 y + \tilde{\omega}_2 t + \tilde{\nu}_2 \tau} + \Gamma_1(k_2) s_2 e^{k_2 x + q_2 y + \omega_2 t + \nu_2 \tau}, \tag{89}$$
$$\psi_3^{(3)} = \Gamma_1(k_3) s_3 e^{k_3 x + q_3 y + \omega_3 t + \nu_3 \tau}, \tag{90}$$
$$\psi_3^{(4)} = \Gamma_1(k_4) s_4 e^{k_4 x + q_4 y + \omega_4 t + \nu_4 \tau} + \Gamma_1(k_5) s_5 e^{k_5 x + q_5 y + \omega_5 t + \nu_5 \tau}, \tag{91}$$
$$\psi_4^{(1)} = p_1 r_4^{(1)} e^{\tilde{k}_1 x + \tilde{q}_1 y + \tilde{\omega}_1 t + \tilde{\nu}_1 \tau} + \Gamma_2(k_1) s_1 e^{k_1 x + q_1 y + \omega_1 t + \nu_1 \tau}, \tag{92}$$
$$\psi_4^{(2)} = p_2 r_4^{(2)} e^{\tilde{k}_2 x + \tilde{q}_2 y + \tilde{\omega}_2 t + \tilde{\nu}_2 \tau} + \Gamma_2(k_2) s_2 e^{k_2 x + q_2 y + \omega_2 t + \nu_2 \tau}, \tag{93}$$
$$\psi_4^{(3)} = \Gamma_2(k_3) s_3 e^{k_3 x + q_3 y + \omega_3 t + \nu_3 \tau}, \tag{94}$$
$$\psi_4^{(4)} = \Gamma_2(k_4) s_4 e^{k_4 x + q_4 y + \omega_4 t + \nu_4 \tau} + \Gamma_2(k_5) s_5 e^{k_5 x + q_5 y + \omega_5 t + \nu_5 \tau}. \tag{95}$$

where $c^{(n)} \equiv c^{(n)}(k_n)$, $c_1^{(n)} \equiv c_1^{(n)}(k_n)$ are given by the eqs.(46,47) with $c_2^{(n)} = 1$; $r_4^{(n)}$ are given by the eq.(74) with $r_3^{(n)} = 1$; parameters $k_n, q_n, \omega_n, \nu_n$ are mutually related by the eqs.(48-50); relationship among parameters $\tilde{k}_n, \tilde{q}_n, \tilde{\omega}_n, \tilde{\nu}_n$ is given by the eqs.(75-77); Γ_n are given by the eqs.(79); s_n and p_n are arbitrary parameters.

General expressions for the functions u_1 and u_2 are very complicated. As an example, let us fix the following list of parameters:

$$k_1 = 1, \ k_2 = -1, \ k_3 = 2, \ k_4 = -2, \ k_5 = 0, \tag{96}$$
$$\beta_1 = 1, \ \beta_2 = -3, \ \gamma_1 = \gamma_2 = \delta_{12} = 1, \tag{97}$$
$$\alpha_{12} = \alpha_{32} = \alpha_{53} = \alpha_{62} = \alpha_{74} = 1, \tag{98}$$
$$\alpha_{54} = \alpha_{84} = 2, \ \alpha_{64} = -1. \tag{99}$$

Then, formulas (41) read:

$$u_1 = \frac{45}{\tilde{\triangle}} \left(s_5 e^{2(x+12\tau)} - 2 s_4 e^{2(3y+4t)} \right) \times \tag{100}$$

$$\left(343s_2e^{\frac{6}{7}y+\frac{48}{49}t}+76p_2e^{\frac{2}{7}x+\frac{144}{49}\tau}\right),$$

$$u_2 = \frac{6}{\tilde{\triangle}}\left(15680s_2s_4e^{\frac{48}{7}y+\frac{440}{49}t}+8967s_2s_5e^{2x+\frac{6}{7}y+\frac{48}{49}t+24\tau}+\right. \tag{101}$$
$$\left.4560s_4p_2e^{\frac{2}{7}x+6y+8t+\frac{144}{49}\tau}+1140s_5p_2e^{\frac{16}{7}x+\frac{1320}{49}\tau}\right),$$
$$\tilde{\triangle}=24010s_2s_4e^{\frac{48}{7}y+\frac{440}{49}t}+21609s_2s_5e^{2x+\frac{6}{7}y+\frac{48}{49}t+24\tau}+$$
$$6840s_4p_2e^{\frac{2}{7}x+6y+8t+\frac{144}{49}\tau}+3420s_5p_2e^{\frac{16}{7}x+\frac{1320}{49}\tau}.$$

This solution has no singularities if

$$\mathrm{sign}(s_2s_4)=\mathrm{sign}(s_2s_5)=\mathrm{sign}(s_4p_2)=\mathrm{sign}(s_5p_2) \tag{102}$$

The nonlinear system keeps the same form (63,64). The change of variables (65) is not effective in this case, so that the nonlinear system is essentially (2+2)-dimensional. If one of parameters x, y, t or τ is fixed, then functions u_1 and u_2 represent (2+1)-dimensional inelastic 3-kinks interaction.

4 Conclusions

The represented method is another way of using the algebraic system of equations for analysis of nonlinear PDE. In comparison with method considered in [30, 31, 32], it supplies wide class of particular solutions for nonlinear PDE which may be solved by this method. In general, nonlinear PDE considered in this paper, are not completely integrable in classical sence. For instance, attempt to introduce commuting flows by the standard way leads, generally speaking, to new constraints on the manifold of available solutions to nonlinear PDE.

There is no formal restriction on the linear differential operators L_1 and L_2 (see eqs.(20)). This makes the algorithm fluxible, but its application field is not defined yet. An interesting problem is construction of commuting flows on submanifold of particular solutions to the given nonlinear PDE.

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